

# Divergent Series: An Arithmetic Approach

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## ABSTRACT

Infinite divergent series can generate some striking results but have been controversial for centuries. The standard approaches of limits and methods of summation have drawbacks which do not account for the full range of behavior of these series. A simpler approach, the arithmetic approach, is developed, which better accounts for divergent series and their sums.

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## INTRODUCTION

[I]n the early years of this century [the 20th] the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and about the present title [*Divergent Series*], now colourless, there hung an aroma of paradox and audacity.

J. E. Littlewood in his preface to [8]

In this paper, we will see that we can find finite sums for infinite divergent series. Surprisingly, we find that it is controversial among mathematicians as to whether these series have sums. We will examine this topic in some detail, and we will develop an approach which enables us to better appreciate these series and the surprising nature of the infinite which they show.

A finite geometric series

$$\sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n$$

can easily be summed by through a recurrence formula. We let  $x$  be the sum:

$$x = \sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n,$$

and then we multiply both sides by  $a$ :

$$xa = a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots + a^{n+1},$$

and then we observe that  $xa + a^m = x + a^{n+1}$ . We therefore have  $xa - x = a^{n+1} - a^m$ , or alternatively  $x - xa = a^m - a^{n+1}$ , which yields

$$x = \frac{a^{n+1} - a^m}{a - 1} = \frac{a^m - a^{n+1}}{1 - a},$$

the well-known result

$$\sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n = \frac{a^m - a^{n+1}}{1 - a}. \quad (1)$$

It may come as a surprise, although it is well known among mathematicians, that an infinite geometric series can also be summed in this way. If we have

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots,$$

then again we set

$$x = \sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots,$$

again multiply both sides by  $a$ ,

$$xa = a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots,$$

and observe that  $xa + a^m = x$ , yielding  $x - xa = a^m$ , or

$$x = \frac{a^m}{1-a},$$

and another well-known result

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = \frac{a^m}{1-a}. \quad (2)$$

If  $m = 0$ , that is, if the first term is 1, then this result becomes

$$\sum_{k=0}^{\infty} a^k = 1 + a^2 + a^3 + a^4 + \dots = \frac{1}{1-a}. \quad (3)$$

For example, if  $a = \frac{1}{2}$  and  $m = 1$ , then the infinite geometric series is  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ . Equation (2) says that this sum is  $\frac{1/2}{1-1/2} = 1$ , that is

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1. \quad (4)$$

Figure 1 shows a visualization of this sum, and the formula seems to agree with what we observe in such a diagram.

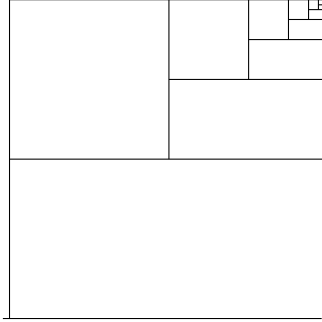


FIG. 1: Diagram of  $1/2 + 1/4 + 1/8 + 1/16 + \dots = 1$

It has long been noticed that, when derived this way, equation (2) seems to be true regardless of the magnitude of  $a$ , except possibly  $a = 1$ . For example, if  $a = 2$ , we could conclude that

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + \dots = -1. \quad (5)$$

This may come as an even bigger surprise than Equation (4). Naturally, it might be wondered how Equation (5) could be true. The intermediate results,  $1$ ,  $1 + 2 = 3$ ,  $1 + 2 + 4 = 7$ ,  $1 + 2 + 4 + 8 = 15$ , etc., known as *partial sums*, grow without limit and are always positive, whereas in Equation (4), the partial sums  $1$ ,  $1 + \frac{1}{2} = \frac{3}{2}$ ,  $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ ,  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$ , etc., get progressively closer to 2 but never exceed it.

Infinite series such as the one in Equation (4), in which the partial sums approach a fixed number, are known as *convergent*, while those that do not, such as the one in Equation (5), are known as *divergent*.

Among mathematicians, there are two points of view on convergent and divergent infinite series. The conventional point of view is that divergent series are meaningless and have no sum, and only convergent series have a sum. In this view, the number that the partial sums converge to, called the *limit*, is considered as the sum of the infinite series.

The alternative point of view is that divergent series are not automatically meaningless but may have a sum. Following this point of view, a theory of divergent series has been developed. This theory is generally consistent and even has a number of applications. References [1, 6, 8, 10, 13, 14] are some of the important standard works of this theory, with [8] being generally regarded as the most important. In this paper, we will explore some of the results of this theory. In the process, we will develop a

new approach which we hope will resolve some longstanding problems. This will include an extension of arithmetic which will allow us to include even the value of  $a = 1$  in equation (2).

#### SOME RESULTS OF EQUATION (2)

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots = \frac{1}{2}. \quad (6)$$

PROOF. Straightforward application of Equation (2) with  $a = -1$ .

$$\sum_{k=1}^{\infty} (-1)^{k+1} k = 1 - 2 + 3 - 4 + 5 - \dots = \frac{1}{4}. \quad (7)$$

PROOF. Let  $x = 1 - 2 + 3 - 4 + 5 - \dots = 1 - (2 - 3 + 4 - 5 + \dots) = 1 - (1 - 2 + 3 - 4 + \dots) - (1 - 1 + 1 - 1 + \dots) = 1 - x - \frac{1}{2}$ . Then  $2x = \frac{1}{2}$ , and  $x = \frac{1}{4}$ .

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + 5 + \dots = -\frac{1}{12}. \quad (8)$$

PROOF. Let  $x = 1 + 2 + 3 + 4 + 5 + \dots$ . Then  $-3x = x - 4x = (1 + 2 + 3 + 4 + 5 + \dots) - 4(1 + 2 + 3 + 4 + 5 + \dots) = (1 + 2 + 3 + 4 + 5 + \dots) - 2(2 + 4 + 6 + 8 + 10 + \dots) = 1 - 2 + 3 - 4 + 5 - \dots = \frac{1}{4}$ , and  $x = -\frac{1}{12}$ .

$$0.999\dots = 1. \quad (9)$$

PROOF. This is a convergent series which we mention for completeness.  $0.999\dots = 9(0.1 + 0.01 + 0.001 + 0.0001\dots) = 9(10^{-1} + 10^{-2} + 10^{-3} + 10^{-4} + \dots) = 9(0.1^1 + 0.1^2 + 0.1^3 + 0.1^4 + \dots) = 9\sum_{k=1}^{\infty} .01^k = 9\left(\frac{0.1}{1-0.1}\right) = 9\left(\frac{0.1}{0.9}\right) = 9\left(\frac{1}{9}\right) = 1$ .

$$\dots 999 = -1. \quad (10)$$

PROOF. By the notation  $\dots 999$  we mean an infinite series of digits going out to the left, just as the notation  $0.999\dots$  means an infinite series of digits going out to the right. Then  $\dots 999 = 9(1 + 10 + 100 + 1000 + \dots) = 9(10^0 + 10^1 + 10^2 + 10^3 + \dots) = 9\sum_{k=0}^{\infty} 10^k = 9\left(\frac{1}{1-10}\right) = 9\left(-\frac{1}{9}\right) = -1$ .

$$\sum_{k=-\infty}^{\infty} a^k = \dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = 0, \quad a \neq 1. \quad (11)$$

PROOF. We have  $\dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = (a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots) + (\dots + a^{m-1} + a^{m-2} + a^{m-3} + a^{m-4} + \dots) = (a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots) + (\dots + (a^{-1})^{1-m} + (a^{-1})^{2-m} + (a^{-1})^{3-m} + (a^{-1})^{4-m} + \dots) = (a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots) + (\dots + (a^{-1})^{1-m} + (a^{-1})^{2-m} + (a^{-1})^{3-m} + (a^{-1})^{4-m} + \dots) = \frac{a^m}{1-a} + \frac{a^{m-1}}{1-a^{-1}} = \frac{a^m}{1-a} + \frac{a^m a^{-1}}{a(1-a^{-1})} = \frac{a^m}{1-a} + \frac{a^m}{a-1} = \frac{a^m}{1-a} - \frac{a^m}{1-a} = 0.$

Alternatively, if  $x = \dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots$ , then  $xa = \dots + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots = x$ , so  $0 = x - xa = x(1 - a)$ . Then  $x$  must be 0, unless  $a = 1$ .

$$\dots 999.999 \dots = 0. \quad (12)$$

PROOF. We have  $\dots 999.999 \dots = 9(\dots + 1000 + 100 + 10 + 1 + 0.1 + 0.01 + 0.001 + \dots) = 9(\dots + 10^3 + 10^2 + 10^1 + 10^0 + 10^{-1} + 10^{-2} + 10^{-3} + \dots) = 9 \sum_{k=-\infty}^{\infty} 10^k = 9(0) = 0.$

$$\sum_{k=0}^{\infty} e^{kx} = 1 + e^{ix} + e^{2ix} + \dots = \frac{1 + i \cot \frac{x}{2}}{2}. \quad (13)$$

PROOF.  $e^{nix} = (e^{ix})^n$ , so  $1 + e^{ix} + e^{2ix} + \dots = \frac{1}{1 - e^{ix}} = \frac{1}{2} + \frac{i}{2} \left( \frac{2i}{e^{ix} - 1} - 1 \right) = \frac{1}{2} + \frac{i}{2} \cot \frac{x}{2}.$

$$\sum_{k=0}^{\infty} (-1)^k e^{2k+1} = e^{ix} - e^{3ix} + e^{5ix} - \dots = \frac{\sec x}{2}. \quad (14)$$

PROOF.  $e^{ix} - e^{3ix} + e^{5ix} - \dots = \frac{e^{ix}}{1 + e^{2ix}} = \frac{1}{2} \sec x.$

$$\sum_{k=1}^{\infty} \cos kx = \cos x + \cos 2x + \cos 3x + \dots = -\frac{1}{2}. \quad (15)$$

PROOF. From Equation (13),  $\cos x + \cos 2x + \dots = \Re(1 + e^{ix} + e^{2ix} + \dots) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}.$

$$\sum_{k=1}^{\infty} \sin kx = \sin x + \sin 2x + \sin 3x + \dots = \frac{\cot \frac{x}{2}}{2}. \quad (16)$$

PROOF. From Equation (13),  $\sin x + \sin 2x + \dots = \Im(1 + e^{ix} + e^{2ix} + \dots) = \frac{1}{2} \cot \frac{x}{2}.$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cos kx = \cos x - \cos 2x + \cos 3x - \dots = -\frac{1}{2}. \quad (17)$$

PROOF. Starting with Equation (15), we replace  $x$  with  $x + \pi$ . Then  $\cos nx$  remains unchanged when  $n$  is even, because we are adding  $2m\pi$  to  $x$ , where  $n = 2m$  and  $m$  is an integer. But when  $n$  is odd, then  $n = 2m+1$ , and we are adding  $2m\pi + \pi$  to  $x$ , and so  $\cos nx$  becomes  $-\cos nx$ . This yields  $-\cos x + \cos 2x - \cos 3x + \dots = -\frac{1}{2}$ , or  $\cos x - \cos 2x + \cos 3x - \dots = -\frac{1}{2}$ .

$$\sum_{k=1}^{\infty} (-1)^{k+1} \sin kx = \sin x - \sin 2x + \sin 3x - \dots = \frac{\tan \frac{x}{2}}{2}. \quad (18)$$

PROOF. We start with Equation (16) and use the same substitution, replacing  $x$  with  $x + \pi$ . Then  $\sin nx$  remains unchanged for  $n$  even and becomes  $-\sin nx$  for  $n$  odd. In addition,  $\cot \frac{x}{2}$  becomes  $\cot (\frac{x}{2} + \frac{\pi}{2}) = \tan \frac{x}{2}$ .

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \ln 2 \quad (19)$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = -\ln 2 \quad (20)$$

PROOF. Starting with Equation (16), integrating each term from 0 to  $x$  gives  $(\cos x - 1) + \frac{1}{2}(\cos 2x - 1) + \frac{1}{3}(\cos 3x - 1) + \dots = \ln \sin \frac{x}{2} = (\cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x + \dots) - (1 + \frac{1}{2} + \frac{1}{3} + \dots)$ .

For  $x = \pi$ , this gives  $(-1 + \frac{1}{2} - \frac{1}{3} + \dots) - (1 + \frac{1}{2} + \frac{1}{3} + \dots) = 0$ . Letting  $y = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ , then also  $y = -1 + \frac{1}{2} - \frac{1}{3} + \dots$ .

For  $x = \pi/2$ , the integration gives  $(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots) - (1 + \frac{1}{2} + \frac{1}{3} + \dots) = \frac{1}{2}(-1 + \frac{1}{2} - \frac{1}{3} + \dots) - (1 + \frac{1}{2} + \frac{1}{3} + \dots) = -\frac{1}{2} \ln 2 = \frac{1}{2}y - y$ , or  $y = \ln 2$ .

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots = \ln(1-x) \quad (21)$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \ln(1+x) \quad (22)$$

PROOF. From Equation (2), we have  $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$  and  $1 - x + x^2 - x^3 + \dots = \frac{1}{1+x}$ . Integrating these from 0 to  $x$  gives the results, which are known as *Mercator's Series*.

$$\sum_{k=1}^{\infty} \frac{2^k}{k} = \frac{2}{1} + \frac{4}{2} + \frac{8}{3} + \frac{16}{4} + \dots = (2k+1)\pi i, \quad k \text{ an integer.} \quad (23)$$

PROOF. Substitute  $x = 2$  into Equation (21) and use the fact that  $\ln(-1) = (2k + 1)\pi i$ .

$$\sum_{k=0}^{\infty} (-1)^k \cos(2k + 1)x = \cos x - \cos 3x + \cos 5x - \dots = \frac{\sec x}{2}. \quad (24)$$

PROOF. From Equation (14),  $\cos x - \cos 3x + \cos 5x - \dots = \Re(e^{ix} - e^{3ix} + e^{5ix} - \dots) = \frac{1}{2} \sec x$ .

$$\sum_{k=0}^{\infty} (-1)^k \sin(2k + 1)x = \sin x - \sin 3x + \sin 5x + \dots = 0. \quad (25)$$

PROOF. From Equation (14),  $\sin x - \sin 3x + \sin 5x + \dots = \Im(e^{ix} - e^{3ix} + e^{5ix} - \dots) = 0$ .

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} \cos kx &= 1^{2n} \cos x - 2^{2n} \cos 2x + 3^{2n} \cos 3x - \dots \\ &= 0, \quad n = 0, 1, 2, \dots \end{aligned} \quad (26)$$

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} \sin kx &= 1^{2n+1} \sin x - 2^{2n+1} \sin 2x + 3^{2n+1} \sin 3x - \dots \\ &= 0, \quad n = 0, 1, 2, \dots \end{aligned} \quad (27)$$

PROOF. Repeated differentiation,  $n$  times, of Equation (17).

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} \sin kx &= 1^{2n} \sin x - 2^{2n} \sin 2x + 3^{2n} \sin 3x - \dots \\ &= (-1)^n \left( \frac{d}{dx} \right)^{2n} \frac{\tan \frac{x}{2}}{2}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (28)$$

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} \cos kx &= 1^{2n+1} \cos x - 2^{2n+1} \cos 2x + 3^{2n+1} \cos 3x - \dots \\ &= (-1)^n \left( \frac{d}{dx} \right)^{2n+1} \frac{\tan \frac{x}{2}}{2}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (29)$$

PROOF. Repeated differentiation,  $n$  times, of Equation (18).

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} \cos(2k+1)x &= \\ 1^{2n} \cos x - 3^{2n} \cos 3x + 5^{2n} \cos 5x - \dots &= (-1)^n \left( \frac{d}{dx} \right)^{2n} \frac{\sec x}{2}, \\ n = 0, 1, 2, \dots & \quad (30) \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \sin(2k+1)x &= \\ 1^{2n+1} \sin x - 3^{2n+1} \sin 3x + 5^{2n+1} \sin 5x - \dots &= (-1)^n \left( \frac{d}{dx} \right)^{2n+1} \frac{\sec x}{2}, \\ n = 0, 1, 2, \dots & \quad (31) \end{aligned}$$

PROOF. Repeated differentiation,  $n$  times, of Equation (24).

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} \sin(2k+1)x &= \\ 1^{2n} \sin x - 3^{2n} \sin 3x + 5^{2n} \sin 5x - \dots &= 0, \quad n = 0, 1, 2, \dots \quad (31) \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \cos(2k+1)x &= \\ 1^{2n+1} \cos x - 3^{2n+1} \cos 3x + 5^{2n+1} \cos 5x - \dots &= 0, \quad n = 0, 1, 2, \dots \quad (33) \end{aligned}$$

PROOF. Repeated differentiation,  $n$  times, of Equation (25).

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} = 1^{2n} - 2^{2n} + 3^{2n} - \dots = 0, \quad n = 1, 2, 3, \dots \quad (34)$$

PROOF. Substitute  $x = 0$  into Equation (26).

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} = 1^{2n+1} - 3^{2n+1} + 5^{2n+1} - \dots = 0, \quad n = 0, 1, 2, \dots \quad (35)$$

PROOF. Substitute  $x = \frac{\pi}{2}$  into Equation (27).

$$\sum_{k=0}^{\infty} (2k+1)^{2n+1} = 1^{2n+1} + 3^{2n+1} + 5^{2n+1} + \dots = 0, \quad n = 0, 1, 2, \dots \quad (36)$$

PROOF. Substitute  $x = \frac{\pi}{2}$  into Equation (32).

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} = 1^{2n} - 3^{2n} + 5^{2n} - \dots = 0, \quad n = 0, 1, 2, \dots \quad (37)$$

PROOF. Substitute  $x = 0$  into Equation (33).

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} &= 1^{2n+1} - 2^{2n+1} + 3^{2n+1} + \dots \\ &= \frac{2^{2n+2} - 1}{2n+2} B_{2n+2}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (38)$$

PROOF.  $B_k$  stands for the  $k$ -th Bernoulli number, which occurs in the power series  $\tan x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k}-1)}{(2k)!} B_{2k} x^{2k-1}$ .

To prove the above identity, we evaluate  $1^{2n+1} \cos x - 2^{2n+1} \cos 2x + 3^{2n+1} \cos 3x - \dots = (-1)^n \left(\frac{d}{dx}\right)^{2n+1} \frac{1}{2} \tan \frac{x}{2}$  for  $x = 0$ . We begin by computing the power series  $\frac{1}{2} \tan \frac{x}{2} = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k}-1)}{(2k)!} B_{2k} \frac{x^{2k-1}}{2^{2k-1}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k-1}}{(2k)!} B_{2k} x^{2k-1}$ , and we will continue by differentiating this series at  $x = 0$ .

To differentiate this series, we observe that  $\left(\frac{d}{dx}\right)^n x^k = k(k-1)(k-2) \dots (k-n+1) x^{k-n}$ . When  $x = 0$ , this product is nonzero only when  $x^{k-n}$  is constant, i.e. when  $k = n$ , at which time  $\left(\frac{d}{dx}\right)^n x^k = k(k-1)(k-2) \dots 1 = k!$ . This means that all terms of the power series vanish, except for  $k = n$ .

Therefore,  $\left(\frac{d}{dx}\right)^{2n+1} x^{2k-1} = (2n+1)!$  when  $2n+1 = 2k-1$  or  $k = n+1$ , and is zero when  $k \neq n+1$ . We then have  $(-1)^n \left(\frac{d}{dx}\right)^{2n+1} \frac{1}{2} \tan \frac{x}{2} = (-1)^n (-1)^{n+2} \frac{(2^{2n+2}-1)(2n+1)!}{(2n+2)!} B_{2n+2} = \frac{2^{2n+2}-1}{2n+2} B_{2n+2}$ .

Hardy and other older authors state this result in a different form, because they used an older system of indexing the Bernoulli numbers. If we let  $B_k^*$  represent the old system and  $B_k$  the new, then the relation is  $B_{2k} = (-1)^{k-1} B_k^*$ . We then obtain the older statement of the result,  $1^{2n+1} - 2^{2n+1} + 3^{2n+1} - \dots = (-1)^n \frac{2^{2n+2}-1}{2n+2} B_{n+1}^*$ .

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} = 1^{2n} - 3^{2n} + 5^{2n} + \dots = \frac{1}{2} E_{2n}, \quad n = 0, 1, 2, \dots \quad (39)$$

PROOF.  $E_k$  stands for the  $k$ -th Euler number, which occurs in the power series  $\sec x = \sum_{n=0}^{\infty} (-1)^k \frac{1}{(2k)!} E_{2k} x^{2k}$ . To prove our identity, we evaluate  $1^{2n} \cos x - 3^{2n} \cos 2x + 5^{2n} \cos 5x - \dots = (-1)^n \left(\frac{d}{dx}\right)^{2n} \frac{1}{2} \sec x$  for  $x = 0$ .

As before, to differentiate the power series, we use the fact that for  $x = 0$ ,  $\left(\frac{d}{dx}\right)^n x^k = k!$  when  $k = n$ , and is zero when  $k \neq n$ . We then have  $(-1)^n \left(\frac{d}{dx}\right)^{2n} \frac{1}{2} \sec x = (-1)^n (-1)^n \frac{(2n)!}{2(2n)!} E_{2n} = \frac{1}{2} E_{2n}$ .

As with the Bernoulli numbers, there is also an older system of indexing the Euler numbers, which leads to a different form of the result by older authors. Letting  $E_k^*$  represent the old system and  $E_k$  the new, the relation is given by  $E_{2k} = (-1)^k E_k^*$ . The older statement of the result is then  $1^{2n} - 3^{2n} + 5^{2n} + \dots = (-1)^n \frac{1}{2} E_n^*$ .

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}. \quad (40)$$

PROOF. Integrate the negative of Equation (17) from 0 to  $x$  to obtain  $\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots = \frac{1}{2} x$ . Integrate again from 0 to  $x$  to obtain  $(1 - \cos x) - \frac{1}{2^2} (1 - \cos 2x) + \frac{1}{3^2} (1 - \cos 3x) - \dots = \frac{1}{4} x^2$ . Evaluate this at  $x = \pi$ .

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}. \quad (41)$$

PROOF. Let  $y = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ . Then  $\frac{\pi^2}{8} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{1}{2^2} - \frac{1}{4^2} - \dots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{1}{4} (1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots) = y - \frac{1}{4} y = \frac{3}{4} y = \frac{\pi^2}{6}$ .

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}. \quad (42)$$

PROOF.  $1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = (1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots) - (\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \dots) = (1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots) - \frac{1}{2^2} (1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots) = \frac{\pi^2}{8} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{12}$ .

## RESULTS INVOLVING INFINITE PRODUCTS

We now consider various infinite products involving prime numbers. Before this, we will first need to know how to multiply two infinite series, and how to multiply an infinite number of binomials.

$$\begin{aligned}
\left(\sum_{j=1}^m a_j\right) \left(\sum_{k=1}^n b_k\right) &= (a_1 + a_2 + a_3 + \dots + a_m)(b_1 + b_2 + b_3 + \dots + b_n) \\
&= \sum_{j=1}^m \sum_{k=1}^n a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 + \dots + a_1 b_n \\
&\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 + \dots + a_2 b_n \\
&\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 + \dots + a_3 b_n \\
&\quad \vdots \\
&\quad + a_m b_1 + a_m b_2 + a_m b_3 + \dots + a_m b_n \tag{43}
\end{aligned}$$

PROOF. This is the product of two series. Both series are finite, and we simply multiply out each series.

$$\begin{aligned}
\left(\sum_{m=1}^{\infty} a_m\right) \left(\sum_{n=1}^{\infty} b_n\right) &= (a_1 + a_2 + a_3 + \dots)(b_1 + b_2 + b_3 + \dots) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n = a_1 b_1 + a_1 b_2 + a_1 b_3 + \dots \\
&\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 + \dots \\
&\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 + \dots \\
&\quad \vdots \\
&= \sum_{m,n=1}^{\infty} a_m b_n. \tag{44}
\end{aligned}$$

PROOF. This is similar to the previous equation, but with each series now infinite.

$$\begin{aligned}
\prod_{k=1}^n (1 + a_k) &= (1 + a_1)(1 + a_2)(1 + a_3) \dots (1 + a_n) \\
&= 1 + \sum_{k=1}^n b_k = 1 + b_1 + b_2 + b_3 + \dots + b_n,
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \sum_{t=1}^n a_t, \\
b_2 &= \sum_{\substack{t,u=1 \\ t \neq u}}^n a_t a_u, \\
b_3 &= \sum_{\substack{t,u,v=1 \\ t,u,v \text{ distinct}}}^n a_t a_u a_v, \\
&\vdots \\
b_k &= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^n a_{t_1} a_{t_2} a_{t_3} \dots a_{t_k} \\
&= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^n \prod_{j=1}^k a_{t_j}.
\end{aligned} \tag{45}$$

PROOF. This is the product of a finite number of binomials, each with a first term of 1. The number of binomials is finite, and we multiply out their product to obtain 1 plus a sum of  $b$  terms.

For the  $b$  terms,  $b_1$  is the sum of all  $a$ ,  $b_2$  is the sum of the products of any two distinct  $a$ ,  $b_3$  is the sum of the products of any three distinct  $a$ , and so on. Each  $b_k$  term consists of  $k$  factors from the set of the  $a$  terms, with the index of each  $a$  term being distinct.

$$\begin{aligned}
\prod_{k=1}^{\infty} (1 + a_k) &= (1 + a_1)(1 + a_2)(1 + a_3) \dots \\
&= 1 + \sum_{k=1}^{\infty} b_k = 1 + b_1 + b_2 + b_3 + \dots,
\end{aligned}$$

where

$$b_1 = \sum_{t=1}^{\infty} a_t,$$

$$\begin{aligned}
b_2 &= \sum_{\substack{t,u=1 \\ t \neq u}}^{\infty} a_t a_u, \\
b_3 &= \sum_{\substack{t,u,v=1 \\ t,u,v \text{ distinct}}}^n a_t a_u a_v, \\
&\vdots \\
b_k &= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^{\infty} a_{t_1} a_{t_2} a_{t_3} \dots a_{t_k} \\
&= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^{\infty} \prod_{j=1}^k a_{t_j}. \tag{46}
\end{aligned}$$

PROOF. Similar to the previous equation, but with the number of binomials now infinite.

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \frac{p}{p-1} &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \dots \\
&= \frac{1}{\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p}\right)} = \frac{1}{\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \left(1 - \frac{1}{11}\right) \dots} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \tag{47}
\end{aligned}$$

PROOF. The products in the first and second lines are taken over all prime numbers. The first line becomes the second line through two identities:  $\frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$ , and  $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$  or its extended form  $\prod \frac{1}{a} = \frac{1}{\prod a}$ .

To see how the second line becomes the third line, we take the first two primes, 2 and 3. From Equation (2),  $\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ , and  $\frac{1}{1-\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

By Equation (43), when we multiply these two infinite series, we get  $\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{2} + \dots$ , where the denominators in the sum all have as their prime factors powers of 2 and 3 only.

As we continue to multiply each side by  $\frac{1}{1-\frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots$  for successive prime numbers  $p$ , the denominators in the sum have as their prime factors powers of the primes up to  $p$ . When all the prime numbers

have been included in the product, all the integers are included in the denominators in the sum.

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \frac{p^n}{p^n - 1} &= \frac{2^n}{1^n} \cdot \frac{3^n}{2^n} \cdot \frac{5^n}{4^n} \cdot \frac{7^n}{6^n} \cdot \frac{11^n}{10^n} \cdots \\
&= \frac{1}{\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p^n}\right)} = \frac{1}{\left(1 - \frac{1}{2^n}\right) \left(1 - \frac{1}{3^n}\right) \left(1 - \frac{1}{5^n}\right) \left(1 - \frac{1}{7^n}\right) \left(1 - \frac{1}{11^n}\right) \cdots} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots = \zeta(n), \\
& \quad n = \dots, -2, -1, 0, 1, 2, \dots
\end{aligned} \tag{48}$$

PROOF. The proof is similar to the previous equation, except that now we use a constant power of the primes instead of the primes themselves. The last equality is the definition of the Riemann zeta function.

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} (1 + p) &= (1 + 2)(1 + 3)(1 + 5)(1 + 7)(1 + 11) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} k = 1 + 2 + 3 + 5 + 6 + 7 + 10 + \dots, \\
& \quad n = \dots, -2, -1, 0, 1, 2, \dots
\end{aligned} \tag{49}$$

PROOF. The product in the first line is taken over all prime numbers. The sum on the second line is taken over all *squarefree* integers, which are those which contain only single powers of prime factors, and so are not divisible by the square of any integer greater than 1.

By Equation (46), the product on the first line when multiplied out becomes 1, plus the sum of all  $p$ , plus the sum of the products of any two distinct  $p$ , plus the sum of the products of any three distinct  $p$ , and so on. Since each occurrence of  $p$  can be used at most once in a product, each term in the sum is squarefree.

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 + \frac{1}{p}\right) &= \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7}\right) \left(1 + \frac{1}{11}\right) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \dots, \\
n &= \dots, -2, -1, 0, 1, 2, \dots
\end{aligned} \tag{50}$$

PROOF. Similar to the above equation, except that  $p$  is replaced by  $\frac{1}{p}$ .

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 + \frac{1}{p^n}\right) &= \left(1 + \frac{1}{2^n}\right) \left(1 + \frac{1}{3^n}\right) \left(1 + \frac{1}{5^n}\right) \left(1 + \frac{1}{7^n}\right) \left(1 + \frac{1}{11^n}\right) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \dots, \\
n &= \dots, -2, -1, 0, 1, 2, \dots
\end{aligned} \tag{51}$$

PROOF. Similar to the previous equation, except that  $\frac{1}{p}$  is replaced by  $\frac{1}{p^n}$ .

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} (1 - p) &= (1 - 2)(1 - 3)(1 - 5)(1 - 7)(1 - 11) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \pm k = 1 - 2 - 3 - 5 + 6 - 7 + 10 + \dots,
\end{aligned}$$

where  $\pm = \begin{cases} + \\ - \end{cases}$  when  $k$  has an  $\begin{cases} \text{even} \\ \text{odd} \end{cases}$  number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \tag{52}$$

PROOF. Similar to Equation (49), except that  $p$  is replaced by  $-p$ .

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p}\right) &= \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \left(1 - \frac{1}{11}\right) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \pm \frac{1}{k} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} + \dots,
\end{aligned}$$

where  $\pm = \begin{cases} + \\ - \end{cases}$  when  $k$  has an  $\begin{cases} \text{even} \\ \text{odd} \end{cases}$  number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \quad (53)$$

PROOF. Similar to the above equation, except that  $p$  is replaced by  $\frac{1}{p}$ .

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p^n}\right) &= \left(1 - \frac{1}{2^n}\right) \left(1 - \frac{1}{3^n}\right) \left(1 - \frac{1}{5^n}\right) \dots \\
&= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \pm \frac{1}{k^n} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \dots = \frac{1}{\zeta(n)},
\end{aligned}$$

where  $\pm = \begin{cases} + \\ - \end{cases}$  when  $k$  has an  $\begin{cases} \text{even} \\ \text{odd} \end{cases}$  number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \quad (54)$$

PROOF. Similar to the previous equation, except that  $\frac{1}{p}$  is replaced by  $\frac{1}{p^n}$ . The product is the reciprocal of the product in Equation (48), which equals  $\zeta(n)$ .

## RESULTS INVOLVING DIVERGENT INTEGRALS

We use the above results to evaluate divergent or improper integrals.

$$\int_0^{\infty} a^x dx = \frac{-1}{\ln a}. \quad (55)$$

PROOF. For an integer  $n$ , we have  $\int_0^n a^x dx = \sum_{k=1}^n \int_{k-1}^k a^x dx = \frac{1}{\ln a} \sum_{k=1}^n (a^k - a^{k-1}) = \frac{1-1/a}{\ln a} \sum_{k=1}^n a^k$ . For the infinite case,  $\int_0^{\infty} a^x dx = \frac{1-1/a}{\ln a} \sum_{k=1}^{\infty} a^k = \frac{1-1/a}{\ln a} \frac{a}{1-a} = \frac{-1}{\ln a}$ .

$$\int_{-\infty}^{\infty} a^x dx = 0. \quad (56)$$

PROOF. By equation (55),  $\int_{-\infty}^{\infty} a^x dx = \int_0^{\infty} a^x dx + \int_0^{\infty} a^{-x} dx = \int_0^{\infty} a^x dx + \int_0^{\infty} \left(\frac{1}{a}\right)^x dx = \frac{-1}{\ln a} + \frac{-1}{-\ln a} = 0$ .

$$\int_0^{\infty} e^x dx = -1. \quad (57)$$

PROOF. Application of equation (55) for  $a = e$ .

$$\int_0^{\infty} e^{-x} dx = 1. \quad (58)$$

PROOF. Application of equation (56) for  $a = e$  and equation (57). This is a convergent integral and the result agrees with the usual approach.

$$a^{\infty} = 0. \quad (59)$$

PROOF. By integration and equation (55),  $\int_0^{\infty} a^x dx = \frac{1}{\ln a} (a^{\infty} - 1) = \frac{1}{\ln a}$ , so  $a^{\infty} = 0$ . Alternatively, for  $n = \infty$  in equation (1) and by equation (2),  $\sum_m a^k = \frac{a^m - a^{\infty}}{1 - a} = \frac{a^m}{1 - a}$ , again yielding  $a^{\infty} = 0$ .

NOTE. The values  $a = 0$  and  $a = 1$  in this and all the equations above must be carefully examined. This is done below in the section on idempotents.

## A PRACTICAL APPLICATION

It can be argued that the formula  $1 + 2 + 4 + 8 + \dots = -1$  has an application in the way that computers store negative numbers. Almost universally, computers store a signed integer using a convention known as *twos-complement arithmetic*, which uses the highest-order bit as a *sign bit* (0 = non-negative, 1 = negative), and the negative of a positive number is generated by switching all the bits (*ones-complement*) and then adding 1 (twos-complement).

Example: In a 8-bit signed integer, the decimal number 20 has the binary representation 00010100. To get -20, we transform the 0's into 1's and the 1's into 0's to get a ones-complement representation of 11101011, and then we add 1 to get a twos-complement representation of 11101100.

The great advantage of the twos-complement system is that the same rules can be used for arithmetic operations on both positive and negative numbers. This in turn helps to make processors faster. Another advantage is that the highest-order bit can be interpreted either as a sign bit or as the negative of the highest power of 2, and the arithmetic is the same.

In the same system, 8-bit signed integers, the twos-complement representation of -1 is 11111111. No matter how many bits there are, the

two's-complement representation of  $-1$  is always all 1's. Since the value of each bit is a power of 2, this is the sum of all the powers of 2 that are available in the computer representation of the number, except that the sign bit is the negative of the highest power. If we increase the number of bits, the sign bit moves to higher powers of 2.

In the theoretical case of an infinite number of bits, the sign bit disappears, and we are left with an infinite number of ones, which represent the number  $-1$ . This is identical to the divergent series formula  $1 + 2 + 4 + 8 + \dots = -1$ .

### THREE APPROACHES

We will consider the following three approaches to divergent series.

1. **Limits.** In this approach, any infinite series is considered to be the limit of its partial sums:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

If the limit exists, the series is said to be convergent, and otherwise divergent. A divergent series has no limit and therefore, strictly speaking, no sum. More informally we may say that the sum is  $\infty$ , but this symbol is usually not defined as a number.

2. **Methods.** In this approach, a convergent series is still considered to be a limit, but a divergent series may be said to have a finite sum in a restricted sense. It has this sum if some calculational procedure based on the terms of the series yields a finite sum. The procedure is called a *method of summation*. Ideally, a method applied to a convergent series should yield the same sum as the limit, and when applied to at least some divergent series may yield a finite sum. Since the sum of a divergent series may vary from one method to another, the equality of the series with its sum is said to exist only in the sense of the method.
3. **Arithmetic.** In this approach, any infinite series is considered simply as a purely infinite arithmetic sum, without regard to a limit. It regards infinite sums as an extension of finite arithmetic.

The limits approach was developed in the early nineteenth century and became nearly universal as direct calculation with the infinite and the infinitesimal was replaced with limits and set theoretical notions. Today, most professional mathematicians are unaware of any alternatives.

The methods approach developed in parallel to the limit approach but remained much less popular. It is today known as the theory of divergent series. By far the most comprehensive treatment of this approach is [8], now considered the standard reference.

The arithmetic approach is the one we develop in this paper. As we will see later, this is not a new approach, but only a better understanding of an old approach.

Hardy [8, p. 6–7] defines the notation  $\sum a_m = s (P)$  to mean that the series  $\sum a_m$  has a sum  $s$  in the sense of a method  $P$ . This symbolism means that equality holds only in a certain sense, because changing the sense or the method of determining the sum can change the value  $s$  that we define as the sum. This weakens the meaning of equality, since it is relative to a method of computation, and not, as is normally the case, independent of it.

The methods approach has several weaknesses:

1. An unworkable conception of weak equality.
2. The failure of all methods, except the Euler continuation method, to account for Equation (2) or any other significant finding.
3. A faulty understanding of the Euler continuation method, including a circular definition in [8], and failure to realize this method as an extension of arithmetic.

We further examine these points in later sections and show how the arithmetic approach avoids all of them.

## OTHER APPROACHES

A few other approaches to divergent series deserve consideration.

**Transformation to convergent series.** In this approach, a divergent series is considered as an encoded form of a convergent series, using the transformation

$$\frac{1}{1-a} = \left(-\frac{1}{a}\right) \frac{1}{1-\frac{1}{a}}.$$

This approach has the disadvantage of conceptualizing simple arithmetic statements as something different from what they state, and therefore of introducing extra steps of calculation. Another disadvantage is that an expression which combines convergent and divergent series, such as  $\sum_{-\infty}^{+\infty} a^n$ , must be considered as the sum of two separate series.

**Congruence.** In this approach, we observe that the partial sums  $\sum_0^j 2^n$  are each congruent to  $-1 \pmod{2^{j+1}}$ . The drawback of this approach is that it requires us to define an equality through a congruence. It also is not immediately clear how to extend it to any other case than  $\sum_0^j 2^n$ , since for integral bases greater than two, the sum is a fraction.

**Adjusted series.** Another approach is to return to the twos-complement arithmetic described above, and generalize its sign bit mechanism.

This seems at first to be a combination of the limit and arithmetic approach, but it actually ends up being essentially just the arithmetic approach.

If we define

$$f(k) = \begin{cases} a^k, & k < n \\ \frac{a^k}{1-a}, & k = n \end{cases}$$

then

$$\sum_{k=0}^n f(k) = \left( \sum_{k=0}^{n-1} a^k \right) + \frac{a^n}{1-a} = \frac{1-a^n}{1-a} + \frac{a^n}{1-a} = \frac{1}{1-a}.$$

This value does not depend on  $n$ —for all values of  $n$ , the sum is constant. In the infinite case, the last term does not exist. Hence we can say that

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

This satisfies the limits approach, because the infinite case is the limit of the finite case. But a limits approach alone does not suggest how to modify the definition of  $\sum_{k=0}^{\infty} a^k$  so that the result is constant. To get the crucial last term, the arithmetic approach is both necessary and sufficient.

## POWER SERIES WITH IDEMPOTENTS

The series

$$x = \sum_{k=0}^{\infty} 0^k = 0 + 0 + 0 + \dots$$

is of fundamental importance, since it is implicit in any infinite series, convergent or divergent, and results whenever we subtract any infinite series from itself. From an arithmetic point of view,  $x = 0 \cdot \infty$ , and so the sum is indeterminate, but according to equation (2),  $x = 0$ , because  $0x = x$ .

We resolve this by taking the approach of Musès that 0 is multivalued. See [11, p. 178–179] and especially [12]. In this view, an expression such as  $x = 0 \cdot \infty$  is indeterminate because 0 is multivalued, and the expression does not specify which values of 0 are involved. We may have two distinct values of 0, which we can call  $0_1$ ,  $0_2$ , etc., and which may have relations such as by  $0_1 = 4 \cdot 0_2$ .

Ordinary equality does not distinguish these multiple values. The sense of equality when we write this type of relation must have the sensitivity to distinguish  $0_1$  from  $0_2$ , so that  $0_1 \neq 0_2$  at this level of sensitivity, whereas at the ordinary level of sensitivity,  $0_1 = 0_2$ , and  $a + 0_1 = a + 0_2 = a$  for any real number  $a$ .

To denote these levels of sensitivity, we use the notation  $a = b [u]$ , where  $u$  is the unit whose multiples we distinguish. In the above example, we write  $0_1 = 0_2 [1]$ ,  $0_1 \neq 0_2 [0_1]$ ,  $0_1 = 4 \cdot 0_2 [0_1]$ , and  $a + 0_1 = a[1]$ . The default unit is 1, so that  $a = b$  means the same as  $a = b [1]$ , unless otherwise stated.

We now examine a power series in  $0_1$  at the  $0_1$  level of sensitivity. For  $x = \sum 0_1^k = 0_1 + 0_1^2 + 0_1^3 + \dots$ , we have  $x = 0_1 + 0_1x$ , and  $0_1 = x - 0_1x$ , and  $x = \frac{0_1}{1-0_1}$ , and  $\frac{0_1}{1-0_1} = 0 [1]$ . Power series in  $0_1$  thus come out to 0, because  $0_1^n$  is smaller than  $0_1$  for  $n > 1$ . The sum  $0_1 + 0_1 + 0_1 + \dots$  is not a power series. The value of  $0 \cdot \infty$  depends on a more precise knowledge of which 0 and which  $\infty$  is being used.

A similar approach can be used on the series

$$x = \sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + \dots$$

Equation (2) gives the value  $x = \frac{1}{0}$ , which is derived from  $x = x + 1$  and  $x - x = x \cdot 0 = 1$ . If we take the view that 0 is multivalued, then  $\frac{1}{0}$  is also multivalued.

## OBJECTIONS

Complete rejection of sums of divergent series results in the limits approach. From the methods approach, the following objections have historically been raised, without complete resolution.

Hardy [8, p. 16] points out several examples of divergent series which appear to give multiple values. For example,

$$x + (2x^2 - x) + (3x^3 - 2x^2) + (4x^4 - 3x^3) + \dots = 0$$

and

$$x + (3x^2 - x) + (7x^4 - 3x^2) + (15x^8 - 7x^4) + \dots = 0$$

give  $1 + 1 + 1 + \dots = 0$  and  $1 + 2 + 4 + \dots = 0$ . But he does not recognize that

$$\begin{aligned} & x + (2x^2 - x) + (3x^3 - 2x^2) + (4x^4 - 3x^3) + \dots \\ &= (x - x) + (2x^2 - 2x^2) + (3x^3 - 3x^3) + (4x^4 - 4x^4) + \dots \\ &= 0 + 0 + 0 + \dots = 0 \times \infty, \end{aligned}$$

which of course is indeterminate, not merely zero.

Using Equation (2), we have shown that we can calculate that  $1 - 1 + 1 - 1 + \dots = \frac{1}{2}$ . It is frequently objected that this cannot be the only correct result, since we could also have  $1 - 1 + 1 - 1 + \dots = (1 - 1) + (1 - 1) + \dots = 0 + 0 + \dots = 0$ . This objection comes from one or both of two points of view:

(1) an infinite sum is a limit; and/or (2) the sum of an infinite number of zeros is zero. The first point of view overlooks the important fact that the type of sum we are considering here is not a limit. The second point of view overlooks the fact that an infinite sum of zeros is  $\infty \times 0$ , which is indeterminate.

Euler was the first to systematically use Equation (2) for divergent series and was the first to state this case of it. Some time afterwards, Callet challenged this view, citing the series

$$1 - a^2 + a^3 - a^5 + a^6 - a^7 + a^8 - \dots$$

Setting  $x$  to this series, we have

$$\begin{aligned} x &= 1 - a^2 + a^3 - a^5 + a^6 - a^8 + a^9 - \dots \\ &= 1 - a^2 + a^3(1 - a^2 + a^3 - a^5 + a^6 - \dots) \\ &= 1 - a^2 + a^3x. \end{aligned}$$

Then  $x - a^3x = 1 - a^2$ , and

$$x = \frac{1 - a^2}{1 - a^3} = \frac{1 + a}{1 + a + a^2}.$$

When  $a = 1$ , we would then have

$$1 - 1 + 1 - 1 + \dots = \frac{2}{3}.$$

Lagrange replied that actually

$$\frac{1 - a^2}{1 - a^3} = 1 + 0a - a^2 + a^3 + 0a^4 - a^5 + a^6 + 0a^7 - a^8 + a^9 + \dots,$$

which for  $a = 1$  becomes

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + 1 + \dots = \frac{2}{3}.$$

This differs from  $1 - 1 + 1 - 1 + \dots$  by the addition of an infinite number of zeroes, which, as we have seen above, is not zero but indeterminate.

Besides this, for  $a = 1$ ,  $\frac{1+a}{1+a+a^2} = \frac{2}{3}$ , but  $\frac{1-a^2}{1-a^3}$  is indeterminate.

We have shown above that  $1 + 2 + 4 + 8 + \dots = -1$ . Hardy [8, p. 15] challenges this as an exclusive result. Starting from the identity

$$\frac{1}{x-1} = \frac{1}{x+1} + \frac{2}{x^2-1},$$

repeated application yields

$$\frac{2}{x^2-1} = \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \frac{16}{x^{16}+1} \dots$$

Hardy states that when  $x = 1$ , this becomes  $1 + 2 + 4 + 8 + \dots = \infty$ , but he does not seem to notice that the original identity is not valid for  $x = 1$ . This is true even in equinfinitesimal arithmetic. It depends on the identity  $x + 1 = \frac{x^2-1}{x-1}$ , but in equinfinitesimal arithmetic  $\frac{1^2-1}{1-1}x = 1$  may be indeterminate or assume any single value, because  $1 - 1$  is indeterminate at the infinitesimal level.

Hardy [8, p. 16] also states that  $1 + 1 + 1 + \dots = -\frac{1}{2}$ , because the well-known Riemann zeta function, which he states as

$$\zeta(x) = \sum_{k=1}^{\infty} k^x,$$

has the value  $\zeta(0) = -\frac{1}{2}$ . The problem with this value is that it is calculated from the definition

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du,$$

which is equivalent to the first expression for  $x > 1$ , but not elsewhere.

## THE THEORY OF DIVERGENT SERIES

For several centuries there been a theory of divergent series, which attempts to show how and why we can find a sum for many types of divergent series. It is generally agreed that the theory of divergent series started with Euler, whose findings on the subject are probably best represented in [5]. This theory has attracted other very well known names, including Poisson, Abel, and Hardy. Hardy's posthumous book [8] of 1949 is generally considered a standard work.

Euler and other mathematicians of the 18th century generally either took a basically arithmetic approach to divergent series, or rejected such series, or vacillated. In the 19th century, the limits approach developed and gradually came to dominate the approach to infinite arithmetic. In the 20th century, the methods approach developed and attracted some attention, but it remained poorly known and never came to dominate over the limits approach.

In recent years, the theory seems to be once again attracting some attention. One example of this transition is the Encyclopedic Dictionary of Mathematics, a standard general mathematics reference work. In the first edition [2], published in 1977, the article *Summability* describes the methods approach to divergent series, while the article *Infinite series* gives only the conventional limits approach and does not even reference the summability article, and the index does not reference the summability article under *series*, only under *summability*. In the second edition [4] of 1987, the summability material has been merged into the it Infinite series article.

Both [3] and [4], in their succinct but legalistic way, define a method as a linear transformation and therefore omit the Euler continuation method. They also use weak equality, the assertion that that equality established through a method is relative to the method.

Hardy does not define method but does define and use weak equality. Hardy additionally admits the Euler continuation method, as described below. This method is best suited to power series, which is the key to the whole theory.

## METHODS OF SUMMATION

Hardy [8] defines several dozen methods of summation. We will now examine a significant cross section of these methods, and a few that have been developed since [8] was published. This includes all the methods commonly encountered in the literature and many other more obscure methods. We will apply each one on the series

$$\sum_{k=0}^{\infty} a^k, \quad |a| > 1. \quad (60)$$

Surprisingly, we will find that only *one* of these methods is capable of summing this series. That method is the *Euler continuation method*, which we examine in detail in the next section.

A key criterion for assessing the validity of any method is whether it is *regular*. Most, but not all, methods defined in [8] are regular. A method is P regular if, for any convergent series  $\sum a_m$ ,  $\sum_{(P)} a_m = \lim \sum a_m$ , i.e. if the method sums any convergent series to its ordinary value as a limit. A method need not be regular, but it is advantageous if it is.

For each of these methods:

- We give a page reference to [8], except for two methods which are not in [8].
- We indicate whether the method is regular.
- We give a definition of the method when it is fairly simple. Otherwise, we express our conclusions in the same notation that [8] uses.
- We apply each method to Equation (60).
- We denote partial sums as  $s_n \equiv \sum^n a_k$ .

**Euler method (E)**, which we call the **Euler continuation method**: [8, p. 7]. See below for a full discussion. Regular. Sums (60) to  $\frac{a^m}{1-a}$ .

**Cesàro mean (C, 1)**: [8, p. 7]. Defined as  $\sum_{(C,1)} a_n = \lim_{n \rightarrow \infty} \frac{s_0 + s_1 + s_2 + \dots + s_n}{n+1}$ . Regular. Diverges for (60).

**Abel sum (A)**: [8, p. 7]. Defined as  $\sum_{(A)} a_n = \lim_{x \rightarrow 1^-} \sum a_n x^n$  if  $\sum a_n x^n$  is convergent for  $0 \leq x < 1$ . Regular. Does not apply to (60) since  $\sum a_n x^n = \sum a^n x^n = \sum (ax)^n$  is not convergent for all  $x \in [0, 1)$ .

**Euler's polynomial method (E, 1):** [8, p. 7]. Regular. Diverges for (60), since  $b_n = (a + 1)^n > 2^n$ .

**Hutton's method (Hu, k):** [8, p. 21]. Regular. A limit of positive terms for (60), and thus diverges.

**Ramanujan's method ( $\mathfrak{R}$ , a):** [8, p. 327]. Regularity not stated in [8]. Does not apply to (60) since  $\lim_{x \rightarrow 0} \left(\frac{d}{dx}\right)^k x^n \neq 0$  for  $k > n$ .

**Borel integral method (B'):** [8, p. 83]. Regular. Sums (60) to  $\frac{1}{1-a}$  only when  $\Re a < 1$ .

**Borel exponential method (B):** [8, p. 80]. Regular. Sums (60) to  $\frac{1}{1-a}$  only when  $\Re a < 1$ .

**Nörlund means (N,  $p_n$ ):** [8, p. 64]. Scheme requiring choice of  $\{p_n\}$ . Regular for some  $\{p_n\}$ . The partial quotients for (60)  $t_m$  are always positive, and thus cannot yield  $\frac{1}{1-a}$ , which is negative.

**Abel means (A,  $\lambda_n$ ):** [8, p. 71]. Scheme requiring choice of  $\{\lambda_n\}$ . Regular. For (60), yields only positive terms, so the method yields a positive limit or no limit.

**Lindelöf method (L):** [8, p. 77]. Regular. Same as  $(A, n \ln n)$  for (60), which fails in all cases.

**Mittag-Leffler method (M):** [8, p. 79]. Regularity not stated in [8]. Similar to Lindelöf method. Hardy shows that  $L$  and  $M$  methods take (60) to  $\frac{1}{1-a}$ , but only on a region  $\Delta$  in the complex plane, called the *Mittag-Leffler star* of  $a^n$  for  $a \in \mathbb{C}$ . This region does not include any point in  $(1, \infty)$ .

**Riemann method (R, k):** [8, p. 89]. Regular for  $k > 1$ . For (60),  $a^n$  eventually overwhelms  $\left(\frac{\sin nh}{nh}\right)^k$ , so the limit diverges.

**Euler's general polynomial method (E, q):** [8, p. 178]. Regular. Requires  $q > 0$  and sums (60) to  $\frac{1}{1-a}$  only within a circle with center at  $-q$  and radius  $q + 1$ , which excludes  $a > 1$ .

**General Cesàro means (C, k):** [8, p. 96]. Regular for  $k > 0$ . Limit of the quotient of two positive terms for (60), diverges.

**Hölder means (H, k):** [8, p. 94]. Regularity not stated in [8]. Limit of positive terms for (60), diverges.

**Ingham's method (I):** [8, p. 376]. Not regular. Limit of the sum of positive terms for (60), diverges.

**Second Nörlund method ( $\bar{N}$ ,  $p_n$ ):** [8, p. 57]. Regular. Yields a positive fraction for (60) and any set of positive  $\{p_n\}$ , and thus cannot yield  $\frac{1}{1-a}$ , which is negative.

**De la Vallée-Poussin's method (VP):** [8, p. 88]. Regular. Limit of the sum of positive terms for (60), which is never negative.

**Bernoulli summability (Be):** Not in [8]. Defined by  $\sum_{(\text{Be})} a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N s_n \binom{N}{n} p^n (1-p)^{N-n}$ , where  $s_n = \sum^n a_k$  and  $0 < p < 1$ . Always yields a positive result for (60) and thus cannot assume a negative value.

**Dirichlet summability (D):** Not in [8]. Defined by  $\sum_{(D)} a_n = \lim_{x \rightarrow 1^+} \sum_{n=1}^{\infty} \frac{s_n}{n^x}$ , where  $s_n = \sum^k a_k$ . Always yields a positive result for (60) and thus cannot assume a negative value.

## THE EULER CONTINUATION METHOD OF SUMMATION

We now examine in detail the Euler continuation method of summation, which Hardy denotes as  $\mathfrak{E}$ . We have seen how every other method commonly used in the current theory of divergent series fails to derive  $\sum a^n = \frac{1}{1-a}$  for  $|a| \geq 1$ . Strangely, even though this is the only such method that can derive this sum, it is usually omitted from works on divergent series. For instance, [4, p. 1415] defines a method as a linear transformation and thus defines away this method.

We will see that this method is poorly understood. We will develop a better understanding of it and see how this yields a satisfactory theory of divergent series. We start with Hardy's definition [8, p. 7]:

If  $\sum a_n x^n$  is convergent for small  $x$ , and defines a function  $f(x)$  of the complex variable  $x$ , one-valued and regular in an open and connected region containing the origin and the point  $x = 1$ ; and  $f(1) = s$ ; then we call  $s$  the  $\mathfrak{E}$  sum of  $\sum a_n$ . The value of  $s$  may naturally depend on the region chosen.

Unfortunately, this definition is circular. If the series defines  $f$ , then  $f(x) = \sum a_n x^n$  within some region, and if this region contains  $x = 1$ , then  $f(1) = \sum a_n$  by definition. This does not extend the definition of the series from the convergent case to the divergent case, but instead assumes that we already have a definition of the divergent case.

Hardy also states [8, p. 10]:

In fact [the Euler continuation method] is not even regular, since  $f(x)$  need not be regular at  $x = 1$  when  $\sum a_n$  converges.

It is even harder to tell what Hardy meant by this statement, since he does not define what it means for a *function* to be regular.

It may be that Hardy meant something more by "defining" the function than simply constructing it from the series. Hardy states [8, p. 8] that he is attempting to bring more rigor to Euler's original method, since the level of rigor in Euler's time was generally less than today. Euler summarized his method in a phrase that Hardy quotes [8, p. 8] and that is well known in this field. Here is Euler's original Latin and a translation:

*Summa cujusque seriei est valor expressionis illius finitae, ex cujus evolutione illa series oritur.*

The sum of every series is the value of an expression which is defined by the process from which that series arises.

This definition is quite different from Hardy's. While it is simpler, it unfortunately leaves us a bit in the dark about what kind of "process" might give rise to a series, and whether the sum will be independent of the process.

However, if we think in terms of radius of convergence, i.e. that the function  $\frac{1}{1-a}$  has a series  $\sum a^n$  which is valid within this radius, where the series is convergent, then what we are looking for is a way of extending the validity of the equality of the series and the function outside of the radius of convergence. In this light, perhaps Euler's criterion means that we simply assume that this equality is valid. By "process," Euler may have meant the algebraic steps of transforming  $\sum a^n$  to  $\frac{1}{1-a}$ .

We will take this as the basis of our interpretation of the Euler continuation method. Here is a more expanded version of this interpretation:

If a function is continuous and has continuous derivatives within a region of the complex plane, and if a power series converges to the function within a second region that is contained within the first region, then we assume that the power series represents the function everywhere within the first region.

In other words, we do not reject an algebraic process simply because it deals with a divergent series. If we have found an algebraic process that enables us to compute equality in the convergent case, then we assume that that process is valid in the divergent case also. This method is clearly regular, since it does not involve any transformation of a convergent series.

This method enables us to compute  $\sum a^n = \frac{1}{1-a}$  for all  $a$ , except possibly  $a = 1$ . It is simple and natural, since it involves no limits and no transformation of terms or partial sums. It can calculate the sum of series that no other method can, and there is no known series that any other method can sum that this method cannot. Therefore, we take this as the sole basis of the arithmetic approach. There is no need for a weakened form of equality, since we assume that a divergent series is strictly equal to its arithmetic sum. The results of this approach can be applied without ambiguity, since the equalities it establishes are as strong as all other equalities.

In this approach, if we wish to use other methods of summation, we denote them as modified summation rather than weakened equality. Szász [14, p. 2] uses the notation  $P(\sum a_m) = s$ . This leaves equality strong, but unfortunately this implies that we first compute the value of  $\sum a_m$  and then apply  $P$  after we compute the sum, whereas the real situation is that a method  $P$  operates first on the terms of the series, and then either sums the modified terms or performs some other operation on them. This means that  $P$  is really a modified form of summation. We will use the notation  $\sum_{(P)} a_m = s$  to denote that a method  $P$  sums a series with terms  $a_m$  to the value  $s$ .

#### SUMMARY OF THE ARITHMETIC APPROACH

The arithmetic approach to divergent series developed in this paper can be summarized as follows.

1. Use only strict equality, not any form of weakened equality.
2. Use algebraic transformations to compute the equality of a series with its sum, instead of merely defining the sum as a limit.
3. Denote methods of summation other than the Euler continuation method as modified forms of summation, instead of modified forms of equality.
4. Use equinfinitesimals to determine the sum of an infinite number of zeros, instead of assuming that the sum is always zero.

#### FOUNDATIONS OF THE ARITHMETIC APPROACH

A deeper aspect of the arithmetic approach is the view that numbers and arithmetic stand on their own and do not require definitions to exist. In Although our discussions require that we define our terms, definitions are not a substitute for mathematical truth. The truth of mathematical expressions must ultimately be verified by observation of nature, just as in any other science. In other words, mathematical truth cannot simply be defined into existence.

In this light, we examine Hardy's remarks in [8, p. 5–6] concerning the role of definitions serve in the methods approach to divergent series, which, in the arithmetic view, contain a mixture of both valid and excessive claims about definitions.

[I]t does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit

of definition; it was not natural to them to say, in so many words, ‘by  $X$  we *mean*  $Y$ . There are reservations to be made, ... but it is broadly true to say that mathematicians before Cauchy asked not ‘How shall we *define*  $1 - 1 + 1 - \dots$ ?’ but ‘What *is*  $1 - 1 + 1 - \dots$ ?’, and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal. [Emphasis his.]

While Hardy further acknowledges that the value of  $\frac{1}{2}$  seems “natural,” he says that assigning this as the sole value of the series actually has problems, which he attributes to lack of proper definition. His solution is to use methods of summation, with its concept of weak equality. However, his definition of the Euler continuation method is quite different from Euler’s, and it is circular. He seems to miss its essence as a simple extension of arithmetic.

The arithmetic approach gains support from the recent development of Maharishi Vedic Mathematics, a formulation of ancient Vedic philosophies and technologies of consciousness in mathematical terms [2, 9] by Maharishi Mahesh Yogi, the founder of the Transcendental Meditation program. In this approach, pure consciousness is the basic experience and governing intelligence of life. The technologies of Maharishi Vedic Mathematics, including Transcendental Meditation, give the experience of pure consciousness. Pure consciousness is equated with zero, which is characterized as the *point of infinity* and the *Absolute Number*, the support of all number systems, and thereby all of natural law. Number thereby becomes a natural experience of self-referral available to everyone.

In the author’s experience, Maharishi Vedic Mathematics gives both experience and understanding that makes one at home with the infinite. The infinite sums of the arithmetic approach to divergent series are an example of the surprising and beautiful nature of the infinite.

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APPENDIX: MATHEMATICIANS' OPINIONS ABOUT DIVERGENT SERIES

The theory of divergent series makes startling claims about elementary facts of arithmetic which it often does not prove in a convincing way. It therefore elicits controversy even from well known mathematicians. Here is a sampling.

*Lorsque  $s(n)$ , sans tendre vers une limite, admet une valeur moyenne  $s$  finie et déterminée, nous dirons que la série  $a(0) + a(1) + a(2) + \dots$  est simplement indéterminée, et nous conviendrons de dire que  $s$  est la somme de la série.*

When  $s(n)$ , without tending towards a limit, admits an average value  $s$  which is finite and determinate, we will say that the series  $a(0) + a(1) + a(2) + \dots$  is simply indeterminate, and we will agree to say that  $s$  is the sum of the series.

Cesàro, quoted in [8, p. 8]

*Il résulte de là une classification des séries indéterminées, qui est sans doute incomplète et pas assez naturelle.*

This results in a classification of the indeterminate series, which is undoubtedly incomplete and not natural enough.

Cesàro, quoted in [8, p. 8]

$[1 + 2 + 4 + 8 + \dots = -1]$  has an air of paradox, since it does not seem natural to attribute a negative sum to a series of positive terms.

Hardy, [8, p. 10]

$-1$  and  $\infty$  are the only "natural" sums [of  $1 + 2 + 4 + 8 + \dots$ ].

Hardy, [8, p. 19]

*Porro hoc argumentandi genus, etsi Metaphysicum magis quam Mathematicum videatur, tamen firmum est: et alioqui Canonum Verae Metaphysicae major est usus in Mathesi, In Analysisi, in ipsa Geometria, quam vulgo putatur.*

Again, this kind of argument, although it appears more metaphysically magical than mathematical, nevertheless is well founded; and besides, the true canon of the metaphysics of our forefathers is used in mathematics, in analysis, in its geometry, for ordinary reckoning.

Leibnitz, quoted in [8, p. 14]

*Les géomtres doivent savoir gré au cit. Callet d'avoir appelé leur attention sur l'espece de paradoxe que présentent les séries dont il s'agit, et d'avoir cherché à les prémunir contre l'application des raisonnements métaphysiques aux questions qui, n'étant que de pure analyse, ne peuvent être décidées que par les premeiers principes et les rgles fondamentales du calcul.*

The geometricians must agree with the cited text. Callet has drawn their attention to the species of paradox that present the series as it really is, and to have sought to secure them against the application of metaphysical reasoning on questions which, not being that of pure analysis, can be decided only by first principles and the fundamental rules of calculation.

Lagrange, quoted in [8, p. 14]

*Darüber hat er zwar kein Exempel gegeben, ich glaube aber gewiß zu sein, daß nimmer eben dieselbe series aus der Evolution zweier wirklich verschiedenen expressionum finitorum entstehen könne.*

He has given no example about it; however I believe it is certain that the same series can never develop from the evolution of two really different finite expressions.

Euler, quoted in [8, p. 14]

*Per rationes metaphysicas . . . quibus in analysi acquiescere queamus.*

By metaphysical reasoning . . . which is able to submit to analysis.

Euler, quoted in [8, p. 14]

*Ich glaube, daß jede series einen bestimmten Wert haben müsse. Um aber allen Schwierigkeiten, welche dagegen gemacht worden, zu begegnen, so sollte dieser Wert nicht mit dem Namen der Summe belegt werden, weil man mit deisem Wort gemeiniglich einen solchen Begriff zu verknüpfen pflegt, als wenn die Summe durch eine wirkliche Summierung herausgebracht würde: welche Idee bei den seriebus divergentibus nicht statffindet.*

I believe that every series must have a certain value. However, in order to meet all difficulties which could be made against it, then this value should not be assigned the name of sum, because one tends to commonly link such a term with this word, as if the sum was brought about by a real summation, which idea does not take place with divergent series.

Euler, quoted in [8, p. 15]

*Quamadmodum autem iste dissensus realis videatur, tamen neutra pars ab altera ullius erroris argui potest, quoties in analysi hujusmodi serierum usus occurrit: quod gravi argumento esse debet, neutram partem in errore versari, sed totum dissidium in solis verbis esse positum.*

However, whatever that disagreement seems to give rise to, still neither party can prove any error by the other, as often in analysis this kind of series resists being used; because serious evidence must exist, neither party is in error, but everyone disagrees only in words of expression.

Euler, quoted in [8, p. 15]

*Cette série n'est ni convergente ni divergente et ce n'est qu'en la considérant ainsi que nous la faisons comme la limite d'une série convergente, qu'elle peut avoir une valeur déterminée. . . . Nous admettrons avec Euler que les sommes de ces séries considérées en elles-même n'ont pas de valeurs déterminées; mais nous ajouterons que chacune d'elles a une valeur unique et qu'on peut les employer dans l'analyse, lorsqu'on les regarde comme les limites des séries convergentes, c'est à dire quand*

*on suppose implicitement leurs termes successifs multipliés par les puissances d'une fraction infiniment peu différent de l'unité.*

This series is neither convergent nor divergent, and it is not that by considering it thus that we make it like the limit of a convergent series, that it can have a given value. . . . We will admit with Euler that the sums of these series considered in themselves do not have definite values; but we will add that each one of them has a single value and that one can employ them in analysis, when one looks upon them like the limits of convergent series, this being said when one implicitly supposes their successive terms multiplied by the powers of a fraction differing infinitesimally from unity.

Poisson, quoted in [8, p. 17]

*Pour moi j'avoue que tous les raisonnements et les calculs fondés sur des séries que ne sont pas convergents . . . me paratront toujours trs suspects, même quand les résultats de ces raisonnements s'accorderaient avec des vérités connues d'ailleurs.*

For my part, I acknowledge that all the reasoning and the calculations based on series that are not convergent . . . will always appear very suspect to me, even when the results of this reasoning would agree with truths known elsewhere.

D'Alembert, quoted in [8, p. 17]

*Je mets encore au rang des illusions l'application que Leibniz et Dan. Bernoulli ont faite du calcul des probabilités.*

I still put at the level of illusion the application that Leibniz and Dan. Bernoulli have made of the theory of probability.

Laplace, quoted in [8, p. 17]

[Divergent series is] the only subject yet remaining, of an elementary character, on which serious schism exists among mathematicians as to absolute correctness or incorrectness of results. . . . The moderns seem to me to have made a similar confusion in regard to their rejection of divergent series; meaning sometimes that they cannot safely be used under existing ideas as to their meaning and

origin, sometimes that the mere idea of anyone applying them at all, under any circumstances, is an absurdity. We must admit that many series are such as we cannot safely use, except as means of discovery, the results of which are to be subsequently verified. . . . But to say that what we cannot use no others ever can . . . seems to me a departure from all rules of prudence.

De Morgan, quoted in [8, p. 19]

Would analysis ever have developed as it has done if Euler and others had refused to use  $\sqrt{-1}$ ?

Hardy, [8, p. 19, immediately after the above De Morgan quote]

The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever.

Abel, quoted in [7, p. 170–171]

I must say a few words on the subject of . . . divergent series. . . . It is not easy to get up any enthusiasm after it has been artificially cooled by the wet blankets of rigorists. . . . I have stated the growth of my views about divergent series. . . . I have avoided defining the meaning of equivalence. The definitions will make themselves in time. . . . My first notion of a series was that to have a finite value it must be convergent. . . . A divergent series also, of course, has an infinite value. Solutions of physical problems must always be in finite terms or convergent series, otherwise nonsense is made. . . . Then came a partial removal of ignorant blindness. In some physical problems divergent series are actually used, notably by Stokes, referring to the divergent formula for the oscillating function  $J_n(x)$ . He showed that the error was less than the last term included. . . . Equivalence does not mean identity. . . . But the numerical meaning of divergent series still remains obscure. . . . There will have to be a theory of divergent series, or say a larger theory of functions than the present, including convergent and divergent series in one harmonious whole.

Heaviside, quoted in [8, p. 36]